Upper Bounds for the Number of Hamiltonian Cycles

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Abstract. An upper bound for the number of Hamiltonian cycles of symmetric diagraphs is established first in this paper, which is tighter than the famous Minc's bound and the Brégman's bound. A transformation on graphs is proposed, so that counting the number of Hamiltonian cycles of an undirected graph can be done by counting the number of Hamiltonian cycles of its corresponding symmetric directed graph. In this way, an upper bound for the number of Hamiltonian cycles of undirected graphs is also obtained.

Keywords: Hamiltonian cycle, NP-complete, counting, #P-complete

I. INTRODUCTION

Let $G = (V, E)$ be an undirected graph with a vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. An edge from v_i to v_j is denoted by (v_i, v_j). For simplicity, the vertices are also denoted as $V = \{1, 2, \cdots, n\}$. A Hamiltonian cycle of G is a closed path that visits each of the vertex once and only once. Similarly, if G is a directed graph, a closed directed path which visits each of the vertex once and only once is a Hamiltonian cycle of a directed graph. In this paper, we use the notation $v_1v_2\cdots v_nv_1$ and $(v_1, v_2, \cdots, v_n, v_1)$ to denote Hamiltonian cycles in undirected and directed graphs respectively.

It is well known that the decision problem whether a graph contains a Hamiltonian is NP-complete. Hence, counting the number of Hamiltonian cycles of a graph is a hard problem too. G.A.Dirac[7] shows the existence of Hamiltonian cycles in the undirected graph G of minimum degree at least $(1/2 + \epsilon)n$, where n is the number of vertices in G and $\epsilon > 0$. Counting the number of Hamiltonian cycles in such graphs is still #P-complete[8]. Hence algorithms and analysis are developed for approximating or estimating the number of Hamiltonian cycles in both directed and undirected graphs . The best asymptotic result of the number of Hamiltonian cycles on random graphs is obtained by Janson[11]. N.Alon et. al. show better lower and upper bounds of the maximum number of Hamiltonian cycles in an n-tournament problem [1, 2]. This paper mainly focuses on bounding the number of Hamiltonian cycles. An upper bound of the number of Hamiltonian of arbitrary symmetric directed graph is proposed. We also prove that the number of Hamiltonian cycles of an undirected graph equals half of that of its corresponding symmetric direct graph. Hence, any bound on a directed graph can be directly applied to bound the number of Hamiltonian cycles of an undirected graph.

Our novel bound on a symmetric directed graph is better than one of its natural bounds Minc's bound and tighter than Brégman's bound in many cases, e.g. when out-degrees are bounded by a constant $K \leq 5$. Our proof of the new bound is mainly based on a random algorithm for counting the number of Hamiltonian cycles on a directed graph, which is modified from Rassmussen's algorithm[15]. To apply the result on a symmetric directed graph to the undirected graph, a very simple but useful transformation that transforms counting the number of Hamiltonian cycles of a undirected graph to that of a symmetric directed graph is proposed.

The structure of this paper is as follows. Some nature bounds for the number of Hamiltonian cycles led by matrix permanent are introduced in section II. The Rassmussen's algorithm for counting the number of Hamiltonian cycles is discussed and a modified algorithm is presented in section III. Some fundamental properties of the algorithms are given. A new bound on a symmetric directed graph is presented in section IV. A transformation extending the result in symmetric direct graphs to undirected graphs is established in section V. In this way, upper bounds of the number of Hamiltonian cycles in undirected graphs is also obtained. Some concluding remarks are proposed in section VI.

II. NATURE BOUNDS VIA MATRIX PERMANENT

To establish the results on the number of Hamiltonian cycles, some related concepts and results are introduced. Consider $G = (V, E)$ be a directed graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. In the following

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section, our notations are only related to the directed graph except in section V.

Definition 2.1 Directed graph G is called a symmetric directed graph iff edge $(v_i, v_j) \in E \Rightarrow (v_j, v_i) \in E$, for any $i \neq j$.

Definition 2.2 An 1-factor of a directed graph G is a spanning subgraph of G in which all in-degrees and out-degrees are 1.

An example of the 1-factor is a spanning union of vertex disjoint directed cycles. Let $NH(G)$ and $F(G)$ denote the number of Hamiltonian cycles and the number of 1-factors of a graph G respectively. Since every Hamiltonian cycle is also an 1-factor, therefore

$$NH(G) \leq F(G) \tag{1}$$

The permanent of a matrix $A = (a_{ij})_{n\times n}$ is defined as

$$\mathrm{Per}(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)} \tag{2}$$

where σ goes over all the permutations $\{1, 2, \cdots, n\}$. The adjacency matrix $A = A_G$ of a graph G is an n by n 0-1 matrix. The matrix $A = (a_{ij})$ is defined as $a_{ij} = 1$, if $(v_i, v_j) \in E$; $a_{ij} = 0$, otherwise. Note the diagonal entries of A are all zero, and for any permutation σ, $a_{i\sigma(i)} = 1$, $i = 1, \cdots, n$ iff their corresponding edges in G form an 1-factor of G. Hence,

$$F(G) = \mathrm{Per}(A). \tag{3}$$

Hence any upper bounds on matrix permanent would provide upper bounds for the number of 1-factors and therefore the number of Hamiltonian cycles. For the permanent of a matrix, the following are two famous upper bounds.

We present a similar definition as permanent of a matrix, and we called it Hamilton of a matrix, which is defined as $\mathrm{ham}(A) = a_{11}$, when $n = 1$, and when $n \geq 2$,

$$\mathrm{ham}(A) = \sum_{\{k_2, k_3, \cdots, k_n\}} a_{k_1k_2}a_{k_2k_3}\cdots a_{k_{n-1}k_n}a_{k_nk_1},$$

where $\{k_2, k_3, \cdots, k_n\}$ is over all the permutations of $\{1, 2, \cdots, n\}/\{k_1\}$ and k_1 is any number from the set $\{1, 2, \cdots, n\}$.

Considering the relation between A and graph G, the elements in the set $\{a_{k_1k_2}, a_{k_2k_3}, \cdots a_{k_{n-1}k_n}, a_{k_nk_1}\}$ are all positive iff their corresponding edges in G form a Hamiltonian cycle. Hence, $\mathrm{ham}(A) = NH(G)$.

Theorem 2.1 (Minc's Bound) Let $A = (a_{i,j})$ be an $n \times n$ 0-1 matrix with the row sum r_i, $i = 1, 2, \cdots, n$. Then

$$\mathrm{Per}(A) \leq \prod_{i=1}^{n} \frac{r_i + 1}{2}. \tag{4}$$

Theorem 2.2 (Brégman's Bound) Let $A = (a_{i,j})_{n\times n}$ be an $n \times n$ 0-1 matrix, and r_i denote the number of ones in the row i, $1 \leq i \leq n$. Then

$$\mathrm{Per}(A) \leq \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}}. \tag{5}$$

Due to Stirling formula $n! \leq \sqrt{2\pi n}n^n/e^{n-1/12}$, the bound in Theorem 2.2 is tighter than that in Theorem 2.1. The bound in Theorem 2.2 is conjectured by Minc in 1963 [14] and later proved by Brégman[5]. It plays an essential role in the proof of the conjecture of Szele by N. Alon[2]. From the formula (1)(3)(4)(5), we can naturally obtain an upper bound of the number of Hamiltonian cycles of a directed graph. Particularly it is an upper bound of the number of Hamiltonian cycles of a symmetric directed graph.

III. MODIFIED RASSMUSSEN'S ALGORITHM

In this section, We suppose $G = (V, E)$ be a directed graph with n vertices $\{1, 2, \cdots, n\}$ and $A = A_G = (a_{ij})_{n\times n}$ be the adjacent matrix of the graph G. Let $A(ij)$ be the $(n-1)\times(n-1)$ matrix obtained by removing row i and column j from the matrix A; $A(i, :)$ is row i of the matrix A. For any set S, let $|S|$ be the number of its elements. We now present the algorithm given by Rassmussen in [15].

Algorithm 3.1
inputs: A, an $n \times n$ 0-1 matrix;
outputs: X_A, the estimator of the number of Hamiltonian cycles in G;
step0: Let $p_i = 0$, $i = 1, \cdots, n$;
step1: For $i = 1$ to n
  If $|A(1, :)| = 1$; Set $p_n = a_{11}$, goto step2;
  Else $W = \{j > 1 : a_{1j} = 1\}$;
    If $W = \emptyset$; Set $X_A = 0$;
      Stop;
    Else
      Choose J from W uniformly at random;
      Let $p_i = |W|$;
      Permutate the column 1 and J;
      Let $A = A(11)$;
step2: $X_A = p_1 \times \cdots \times p_n$.

Algorithm 3.1 presents an unbiased estimator of the number of Hamiltonian cycles G, which means the expectation of the output X_a is the number of Hamiltonian cycles in G. We present another point of view of Algorithm 3.1. Through one random experiment of Algorithm 3.1, if the output Y_A is not zero, one obtains a Hamiltonian cycle of G by putting together all the edges corresponding to the selected elements in A. Hence, each Hamiltonian cycle of G can be selected with certain probability. Suppose the Hamiltonian cycle $(1, k_1, \cdots, k_{n-1}, 1)$ has been chosen, where $\{k_i, i = 1, 2, \cdots, n-1\}$ is a permutation of $\{2, 3, \cdots, n\}$. In ith iteration of step1 in Algorithm 3.1, the probability of some element a_{ij} is selected with probability

$$\frac{1}{p_i}, \quad i = 1, \cdots, n.$$

Hence this corresponding Hamiltonian cycle formed by the edges corresponding to the chosen elements is selected with the probability $\frac{1}{p_1} \times \cdots \times \frac{1}{p_n}$ and the output is $p_1 \times \cdots \times p_n$. From this viewpoint, the output is an unbiased estimator of the number of Hamiltonian cycles of G. We state it below.

Theorem 3.1 Let X_A the output of Algorithm 3.1 . Then $E(X_A) = NH(G)$.

Proof: Let $\mathcal{H}(i)$ be one selected Hamiltonian cycle and $Y_{\mathcal{H}(i)}$ denote the output when $\mathcal{H}(i)$ is selected. From the above analysis, we see $\mathcal{H}(i)$ can be selected with probability $\frac{1}{Y_{\mathcal{H}(i)}}$. Hence,

$$E(X_A) = \sum_{i=1}^{NH(G)} Y_{\mathcal{H}(i)} \frac{1}{Y_{\mathcal{H}(i)}} = NH(G)$$

gives the result. □

Note that in one random experiment in Algorithm 3.1 in step1, the element is selected by the ascending order of the row, or equivalently say, we select the first element from the first row of A, then select the second element from the second row of A and so on. If we select the element by another fixed order of the row, we can obtain Algorithm 3.2, which performs an essential part in getting new upper bound in a symmetric directed graph. In the following Algorithm 3.2, the matrix B is used to determine which row are selected at each iteration step1 of Algorithm 3.2 in a random experiment to construct a Hamiltonian cycle. In each independently random experiment running Algorithm 3.2, if B is chosen, it remains unchanged, which promises the results from the independent experiment to be identical random variables.

Algorithm 3.2

inputs: A, an adjacent matrix of graph G;
$B = (b_{ij})_{n\times n}$ a matrix, where b_{ij} is chosen from $\{1, 2, \cdots, n-i+1\}$, for any $i, j \in n$.
outputs: X_A, the estimator of the number of Hamiltonian cycles in G;
step0: Let $p_i = 0$, $i = 1, \cdots, n$ and $k = 1$.
step1: For $i = 1$ to n
  Set $g_i = b_{ik}$;
  If $|A(g_i, :)| = 1$;
    Set $p_n = a_{g_i 1}$, goto step2;
  Else $W = \{j \neq g_i : a_{g_i j} = 1\}$
    If $W = \emptyset$; Set $X_A = 0$; Stop;
    Else
      Choose J from W uniformly at random;
      Let $p_i = |W|$ and $k = J$;
      Permutate the column g_i and J;
      Let $A = A(g_i g_i)$;
step2: $X_A = p_1 \times \cdots \times p_n$.

We now show that Algorithm 3.2 presents an unbiased estimator of the number of Hamiltonian cycles in G. Before present the proof we need a technique lemma.

Lemma 3.1 Let $A'(ij)$ denote the matrix obtained from A by removing row i and column i after permutating column i and column j. Then, if $n \geq 2$

$$\mathrm{ham}(A) = \sum_{j \neq k_1} a_{k_1 j}\, \mathrm{ham}(A'(k_1 j)).$$

Proof: Without loss of generality, we suppose $k_1 = 1$.
The base case $n = 2$ is trivial.
Suppose the case $n-1$ holds.
Now we see the case n. Since

$$\begin{aligned}
\mathrm{ham}(A) &= \sum_{\{k_2,k_3,\cdots,k_n\}} a_{1k_2} a_{k_2k_3} \cdots a_{k_{n-1}k_n} a_{k_n 1} \\
&= \sum_{j=2}^{n} \sum_{\{k_3,\cdots,k_n\}} a_{1j} a_{jk_3} \cdots a_{k_{n-1}k_n} a_{k_n 1} \\
&= \sum_{j=2}^{n} a_{1j} \sum_{\{k_3,\cdots,k_n\}} a_{jk_3} \cdots a_{k_{n-1}k_n} a_{k_n 1}
\end{aligned}$$

where $\{k_3, \cdots, k_n\}$ is over $\{2, \cdots, n\}/\{j\}$.
Hence, we need only to show

$$\sum_{\{k_3,\cdots,k_n\}} a_{jk_3} \cdots a_{k_{n-1}k_n} a_{k_n 1} = \mathrm{ham}(A'(1j))$$

By induction, we know

$$\begin{aligned}
\mathrm{ham}(A'(1j)) &= \sum_{\{k'_2,\cdots,k'_n\}} a'_{k'_2k'_3} \cdots a'_{k'_{n-1}k'_n} a'_{k'_n k'_2} \\
&= \sum_{\{k'_3,\cdots,k'_n\}} a'_{1k'_3} \cdots a'_{k'_{n-1}k'_n} a'_{k'_n 1}
\end{aligned}$$

where $\{k'_3, \cdots, k'_n\}$ is over all the permutations of $\{2, \cdots, n-1\}$. Recall the definition of $A'(1j)$, which is obtained by removing row 1 and column 1 after permutating column 1 and column j, then we know $a'_{1k'_3} = a_{j,k'_3+1}$ and $a'_{k'_3k'_4} = a'_{k'_3+1,k'_4+1}$, $\cdots$, $a'_{k'_{n-1}k'_n} = a'_{k'_{n-1}+1,k'_n+1}$, $a'_{k'_n 1} = a'_{k'_n+1,1}$, which completes the proof. $\square$

By lemma 3.1, it's sufficient to show Algorithm 3.2 presents an unbiased estimator of the number of Hamiltonian cycles.

Theorem 3.2 Let X_A the output of Algorithm 3.1. Then $E(X_A) = NH(G)$.

Proof: We go on to show $E(X_A) = NH(G)$ by induction on n.
The base case $n = 1$ is trivial.
Suppose the case $n-1$ holds, for the case n,

$$\begin{aligned} E(X_A) &= \sum_{j\neq k_1} E(X_A|J=j)P(J=j) \\ &= \sum_{j\neq k_1} E(p_1 X_{A'(k_1 j)})/p_1 \\ &= \sum_{j\neq k_1} E(X_{A'(k_1 j)}). \end{aligned}$$

By induction, $E(X_{A'(k_1j)}) = \mathrm{ham}(A'(k_1j))$ and Lemma 3.1, then

$$E(X_A) = \sum_{j\neq k_1} \mathrm{ham}(A'(k_1j)) = \mathrm{ham}(A) = NH(G).$$

This completes the proof. $\square$

IV. AN UPPER BOUND OF SYMMETRIC DIRECTED GRAPHS

We now present the upper bounds for the symmetric directed graph.

Theorem 4.1 Let G be a symmetric directed graph, $A = A_G = (a_{i,j})_{n\times n}$ be the adjacent matrix of G and r_i denote its sum of row i, $i = 1, 2, \cdots, n$, $n \geq 3$. N denotes the number of Hamiltonian cycles of G. Then

$$N \leq \frac{1}{2^{n-1}} \prod_{i=1}^{n} r_i.$$

Proof: Let $\mathcal{H}(j) = (m_1, m_2, \cdots, m_n, m_1)$ be one of the Hamiltonian cycles of G, where $j = 1, 2, \cdots, N$. m_i ($i = 2, 3, \cdots, n$) is a permutation of $\{1, 2, \cdots, n\}/\{m_1\}$. In Algorithm 3.2, choose $b_{11} = m_1$. In step 1, choose elements by the row order of $m_2, \cdots, m_n$ such that the edges corresponding to the chosen elements constituting $\mathcal{H}(j)$. Let $S_i = \{j : a_{m_i j} = 1\}$ and p_i^H denote the value of ith iteration of step1 in Algorithm 3.2, where $i = 1, \cdots, n$ and $m_{n+1} = m_1$. Then
$p_1^H = r_{m_1}$,
$p_i^H = |Si/\{m_1, m_2, \cdots, m_i\}|$, $i = 2, \cdots, n-1$,
$p_n^H = 1$.
Let $X_{\mathcal{H}(j)}$ be the output when the edges corresponding to the chosen elements form $\mathcal{H}(j)$, then $X_{\mathcal{H}(j)} = \prod\limits_{i=1}^{n} p_i^H$.
Since this is a symmetric directed graph and $n \geq 3$, there exits a Hamiltonian cycle $\mathcal{H}^{'}(j) = (m_1, m_n, m_{n-1}, \cdots, m_1)$ different from $\mathcal{H}(j)$. Let $p_i^{H'}$ be the value of ith iteration of step1 in Algorithm 3.2 and $X_{\mathcal{H}'(j)}$ the output when the edges corresponding to the chosen elements form $\mathcal{H}^{'}(i)$, where $i = 1, \cdots, n$ and $m_0 = m_n$. Then by Algorithm 3.2,
$p_1^{H'} = r_{m_1}$, $p_2^{H'} = 1$,
$p_i^{H'} = |S_i/\{m_1, m_i, m_{i+1}, \cdots, m_n\}|$, $i = 3, \cdots, n$,
$X_{\mathcal{H}'(j)} = \prod\limits_{i=1}^{n} p_i^{H'}$.
Therefore

$$X_{\mathcal{H}(j)} X_{\mathcal{H}'(j)} = \prod_{i=1}^{n} p_i^H p_i^{H'}.$$

Considering the symmetry of A and $a_{m_i m_i} = 0$, $i = 1, 2, \cdots, n$, then $p_i^H + p_i^{H'} \leq r_{m_i}$, $i = 2, \cdots, n$. Hence

$$\begin{aligned}
X_{\mathcal{H}(j)} X_{\mathcal{H}'(j)} &= \prod_{i=1}^{n} p_i^H p_i^{H'} \\
&\leq r_{m_1}^2 \prod_{i=2}^{n} p_i^H (r_{m_i} - p_i^H) \\
&\leq \frac{1}{4^{n-1}} \prod_{i=1}^{n} r_{m_i}^2 \\
&= \frac{1}{4^{n-1}} \prod_{i=1}^{n} r_i^2.
\end{aligned}$$

Let X_A be the output of Algorithm 3.2. Then by Theorem 3.1 and Theorem 3.2 $P(X_A = X_{\mathcal{H}(j)}) = \frac{1}{X_{\mathcal{H}(j)}}$. From Algorithm 3.2, we know the output may be zero with certain probability, hence $\sum\limits_{j=1}^{N} \frac{1}{X_{\mathcal{H}(j)}} \leq 1$. Set $N = NH(G)$, we have

$$\begin{aligned}
N &\leq \frac{N}{\sum\limits_{j=1}^{N} \frac{1}{X_{\mathcal{H}(j)}}} \\
&\leq \sqrt[N]{\prod_{j=1}^{N} X_{\mathcal{H}(j)}} \\
&= \sqrt[2N]{\prod_{j=1}^{N} X_{\mathcal{H}(j)} X_{\mathcal{H}'(j)}} \\
&\leq \sqrt[2N]{\prod_{j=1}^{N} \frac{1}{4^{n-1}} \prod_{i=1}^{n} r_i^2} = \frac{1}{2^{n-1}} \prod_{i=1}^{n} r_i.
\end{aligned}$$

Thus the result follows. □

Theorem 4.2 Let $A = (a_{i,j})_{n\times n}$ be an adjacent matrix of a symmetric directed graph and r_i be the sum of row i of A. Then

$$\frac{1}{2^{n-1}} \prod_{i=1}^{n} r_i \leq \prod_{i=1}^{n} \frac{r_i + 1}{2}.$$

Proof: Since

$$\begin{aligned}
\frac{\prod\limits_{i=1}^{n} \frac{r_i+1}{2}}{\frac{1}{2^{n-1}} \prod\limits_{i=1}^{n} r_i} &= \frac{1}{2}(1 + \frac{1}{r_i})^n \\
&\geq \frac{1}{2}(1 + \frac{1}{n})^n \geq 1.
\end{aligned}$$

Thus the result follows. □

Theorem 4.2 shows our upper bound is tighter than Minc's bound (4). In many cases, the new bound is better than Brégman's bound (5). For example,

$$\frac{1}{2^{n-1}} \prod_{i=1}^{n} r_i \leq \prod_{i=1}^{n} (r_i!)^{1/r_i},$$

when $n \geq 100$, $r_i \leq 5$, $i = 1, 2, \cdots, n$.

V. BOUNDS OF UNDIRECTED GRAPHS

The notations or definitions related to the undirected graphs are only stated in this section. The problem of counting the number of Hamiltonian cycles in an undirected graph is transformed to that of counting the number of Hamiltonian cycles in a symmetric directed graph. This transformation is very simple but powerful. Let G be an undirected graph with vertices $\{1, 2, \cdots, n\}$, where $n \geq 3$. G is a simple graph. Define a symmetric directed graph G' corresponding to G by replacing each edge (i, j) of G with two directed edges (i, j) and (j, i). Let H_G and $H_{G'}$ denote the set of the Hamiltonian cycles in G and G' respectively. $\mathcal{P}(H_{G'})$ denotes the power set of $H_{G'}$. Recall we use the notation $v_1 v_2 \cdots v_n v_1$ and $(v_1, v_2, \cdots, v_n, v_1)$ to denote Hamiltonian cycles in undirected and directed graphs respectively.

Theorem 5.1 Let $\mathcal{H} = m_1 m_2 \cdots m_n m_1$ be a Hamiltonian cycle in H_G. Then there are at least two Hamiltonian cycles $(m_1, m_2, \cdots, m_n, m_1)$ and $(m_1, m_n, m_{n-1}, \cdots, m_1)$ in $H_{G'}$. Define a map φ from H_G to $\mathcal{P}(H_{G'})$ as follows:

$$\varphi(\mathcal{H}) = \{(m_1, m_2, \cdots, m_n, m_1), (m_1, m_n, m_{n-1}, \cdots, m_1)\}.$$

Let $Im\varphi$ denote the image of the map φ and $\mathcal{H}' = m'_1 m'_2 \cdots m'_n m'_1$ be a different Hamiltonian cycles from $\mathcal{H}$. Then

$$\varphi(\mathcal{H}) \cap \varphi(\mathcal{H}') = \emptyset \qquad and \qquad \cup Im\varphi = H_{G'}.$$

Proof: Due to the symmetry of the graph and $n \geq 3$, if there is a Hamiltonian cycle $(m_1, m_2, \cdots, m_n, m_1)$ in $H_{G'}$, there must be a different Hamiltonian cycle $(m_1, m_n, m_{n-1}, \cdots, m_1)$ in $H_{G'}$. These two Hamiltonian cycles obviously has a pre-imagine corresponding to the Hamiltonian cycle $m_1 m_2 \cdots m_n m_1$ in H_G. Note $(m_1, m_2, \cdots, m_n, m_1)$ is in $\varphi(m_1 m_2 \cdots m_n m_1)$. Hence, $\cup Im\varphi \supseteq H_{G'}$. Obviously, $\cup Im\varphi \subseteq H_{G'}$. Therefore

$$\cup Im\varphi = H_{G'}.$$

Suppose there are two different Hamiltonian cycles $\mathcal{H} = m_1 m_2 \cdots m_n m_1$ and $\mathcal{H}' = m'_1 m'_2 \cdots m'_n m'_1$ in H_G. We know that they are different iff there exits a vertex $\{m_i\}$=$\{m'_j\}$ such that at least one of the two neighbor vertices of $\{m_i\}$ is not in the set of two neighbor vertices of $\{m'_j\}$. Hence $(m_1, m_2, \cdots, m_n, m_1)$ is different from $(m'_1, m'_2, \cdots, m'_n, m'_1)$ and $(m'_1, m'_n, m'_{n-1} \cdots, m'_2, m'_1)$, we know $(m_1, m_2, \cdots, m_n, m_1)$ is not in the set $\varphi(\mathcal{H}')$. Similarly, $(m_1, m_n, m_{n-1}, \cdots, m_1)$ is not in $\varphi(\mathcal{H}')$. Then $\varphi(\mathcal{H}) \cap \varphi(\mathcal{H}') = \emptyset$. $\square$

Corollary 5.1 Let $NH(G)$ and $NH(G')$ denote the the number of Hamiltonian cycles in undirected graph G and its corresponding symmetric directed graph G' respectively, then

$$NH(G) = \frac{1}{2} NH(G')$$

Proof: This result is a straightforward deduction of the Theorem 5.1. $\square$

Corollary 5.2 Let G be an undirected graph with vertices $\{1, 2, \cdots, n\}$, $n \geq 3$, $NH(G)$ be the number of Hamiltonian cycles in the graph G. d_i denotes the degree of the vertex $\{i\}$. Then

$$NH(G) \leq \frac{1}{2^{n+1}} \prod_{i=1}^{n} (d_i + 1),$$

$$NH(G) \leq \frac{1}{2^n} \prod_{i=1}^{n} d_i$$

and

$$NH(G) \leq \frac{1}{2} \prod_{i=1}^{n} (d_i!)^{1/d_i}.$$

Proof: By the results of Theorem 2.1, 2.2, 4.1 and Corollary 5.1, this corollary follows. $\square$

VI. CONCLUDING REMARKS

A novel upper bound of the number of Hamiltonian cycles on a symmetric directed graph is presented first in this paper, which is tighter than the famous Minc's bound and better than the bound by Brégmman in many cases.

A transformation from the problem of Hamiltonian cycles of an undirected graph to that of the symmetric directed graph is constructed. Using this transformation and the bounds for directed graphs, upper bounds for the number of Hamiltonian cycles in undirected graph are obtained. The significance of this transformation also lies in the fact that the algorithms for counting the number of Hamiltonian cycles in a directed graph can be directly applied to count the number of Hamiltonian cycles in an undirected graph.

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